Game theoretic approach to skeletally Dugundji and Dugundji spaces

by

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Our motivations

Dugundji spaces were introduced by A. Pełczyński (1968). Their topological description:

A compact Hausdorff space is Dugundji if and only if it has a multiplicative lattice of open maps;

was obtained by E. Shchepin (1976 - 1981). Skeletally Dugundji spaces are skeletal analogue of Dugundji spaces:

A Tychonoff space is skeletally Dugundji if it has a multiplicative lattice of skeletal maps.

Such a class of Tychonoff spaces was first identified by us (i.e., me, A. Kucharski and V. Valov) in (2013). Today, I present the results started two years ago, and I think (i.e., I hope) it will still be continued!!!

Multiplicative lattices

Let $g : X \to g[X]$ and $\phi : X \to \phi[X]$ be two maps. If there exists a map $h : \phi[X] \to g[X]$ such that $g = h \circ \phi$, then we will briefly write $\phi \prec g$. A family Ψ of maps with a common domain which is a Tychonoff X is called a *multiplicative lattice* whenever the following two condition (L1) and (L2) are fulfilled:

(L1) For any map $f : X \to f[X]$ there exists $\phi \in \Psi$ with $\phi \prec f$ and $w(\phi[X]) \leq w(f[X])$; [Here, $w(Y)$ denotes the weight of a space Y.]

(L2) If $\{\phi_\alpha : \alpha \in \mathbb{J}\} \subset \Psi$, then the diagonal (map) $\triangle\{\phi_\alpha : \alpha \in \mathbb{J}\}$ is homeomorphic to some element of Ψ. [Two maps $f : X \to Y$ and $g : X \to Z$ are *homeomorphic* if there is a homeomorphism $h : Z \to Y$ such that $f = h \circ g$.]

(L0) If Ψ consists of open, d-open or skeletal maps, then it is called a *multiplicative lattice of open, d-open or skeletal maps,* respectively.

ω-multiplicative lattice, skeletal and d-open maps

If conditions (L1) and (L2) are fulfilled in respect of maps onto second countable spaces in (L1) (i.e., $w(f[X]) \leq \omega$ is assumed for all $f \in \Phi$) and countable diagonals in (L2) (i.e. it is assumed $|\mathbb{J}| \leq \omega$), then Φ is called an *ω-multiplicative lattice* .

Recall that, in accordance with J. Mioduszewski and L. Rudolf (1969), a map $f : X \to Y$ is said to be *skeletal* (resp., *d-open* in the sense of M. Tkachenko, (1981)) if

$$\operatorname{Int} \overline{f[U]} \neq \varnothing \text{ (resp., } f[U] \subseteq \operatorname{Int} \overline{f[U]})$$

for every nonempty open $U \subseteq X$. Obviously, every d-open map is skeletal. Moreover, any d-open map between compact Hausdorff spaces has to be an open map, Tkachenko (1981).

Let $\mathcal{P}$ be a family of open subsets of a topological space X. For every $x \in X$ consider the set

$$[x]_{\mathcal{P}} = \{y \in X : y \in V \Leftrightarrow x \in V \text{ for all } V \in \mathcal{P}\}.$$

Let $X/\mathcal{P}$ be the family of all classes $[x]_{\mathcal{P}}$ and $q_{\mathcal{P}} : X \to X/\mathcal{P}$ be the map $x \mapsto [x]_{\mathcal{P}}$. The topology on $X/\mathcal{P}$ is generated by the sets $q_{\mathcal{P}}[V] = \{[x]_{\mathcal{P}} : x \in V\}$, where $V \in \mathcal{P}$.

Lemma

Let $\mathcal{P}$ be a family of open subsets of a topological space X such that $X = \bigcup \mathcal{P}$. If $\mathcal{P}$ is closed under finite intersections, then the family $\{q_{\mathcal{P}}[V] : V \in \mathcal{P}\}$ is a base for $X/\mathcal{P}$ and the map $q_{\mathcal{P}}$ is continuous.

Theorem

Let $\mathcal{P} = \langle \bigcup\{\mathcal{P}_\alpha : \alpha \in \Lambda\}\rangle$ with each $\mathcal{P}_\alpha$ being an open cover of X closed under finite intersections and finite unions such that $X/\mathcal{P}_\alpha$ is a Tychonoff space. Then $X/\mathcal{P}$ is also Tychonoff and the map $q_\mathcal{P}$ is homeomorphic to the diagonal map $q = \triangle\{q_{\mathcal{P}_\alpha} : \alpha \in \Lambda\}$.

Proof.

See pages 3-4 in the article by A. Kucharski, Sz. Plewik and V. Valov (2015), which is down-loadable from the arXiv:1501.06734. □

The space $X/\mathcal{P}$ is not always Tychonoff. According to A. Kucharski and Sz. Plewik (2008), the space $X/\mathcal{P}$ is Tychonoff if $\mathcal{P}$ is closed under finite unions and finite intersections and $\mathcal{P}$ has the following property:

Propetry Seq, club families

(Seq) – For every $W \in \mathcal{P}$ there exist sequences

$$\{U_n : 0 \leq n\} \subseteq \mathcal{P} \text{ and } \{V_n : 0 \leq n\} \subseteq \mathcal{P}$$

such that $U_k \subseteq X \setminus V_k \subseteq U_{k+1}$, for each k, and $\bigcup\{U_n : 0 \leq n\} = W$.

According to P. Daniels, K. Kunen and H. Thou (1994), a family $\mathcal{C} \subseteq [\,coZ(X)]^{\omega}$ is said to be a *club* if:

(i) $\mathcal{C}$ is closed under increasing ω-chains, i.e., if $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \ldots$ is an increasing ω-chain from $\mathcal{C}$, then $\bigcup\{\mathcal{A}_n : 0 < n\} \in \mathcal{C}$;

(ii) for any $\mathcal{B} \in [coZ(X)]^{\leq\omega}$ there exists $\mathcal{A} \in \mathcal{C}$ with $\mathcal{B} \subseteq \mathcal{A}$.

A club $\mathcal{C}$ is said to be a *c-club* if it satisfies the following condition:

(iii) – If $\mathcal{P} \in \mathcal{C}$, then $\mathcal{P} \subset_c coZ(X)$, and also $\mathcal{P}$ has the property Seq and $\mathcal{P}$ is closed under finite union and finite intersections.

The relationship $\mathcal{P} \subset_c coZ(X)$ – in other words $\mathcal{P}$ is completely embedded in $coZ(X)$ – means that $\mathcal{P} \subseteq coZ(X)$ and

(c) – For any nonempty $V \in coZ(X)$ there exists $W \in \mathcal{P}$ such that if $U \in \mathcal{P}$ and $U \subseteq W$, then $U \cap V \neq \varnothing$.

However, (c) may be replaced by the following:

(c*) – For any $\mathcal{W} \subseteq \mathcal{P}$, the family $\mathcal{W}$ is predense in $\mathcal{P}$ if and only if $\mathcal{W}$ is predense in $coZ(X)$.

P. Daniels, K. Kunen and H. Thou the above equivalence remained without proof. For those who do not like the concept of predense, we gave details in arXiv:1501.06734.

Additive c-club

Our key notion of additive clubs consisting of co-zero sets was inspired by condition (6) in Theorem 5.3.9 by L. Heindorf and L. Shapiro (1994), which characterizes weakly projective Boolean algebras. This condition was attributed to T. Jech. It allows to expand the study of weakly projective Boolean algebra – or equivalently its Stone spaces – to much wider classes, namely Tychonoff spaces.

Thus: If $\mathcal{C}$ is a c-club and $\langle \mathcal{A}_1 \cup \mathcal{A}_2 \rangle \in \mathcal{C}$ for all $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{C}$, then $\mathcal{C}$ is called an *additive* c-club. Here, $\langle \mathcal{A} \rangle$ denote the least family which contains $\mathcal{A}$ and is closed under finite intersections and finite unions.

Every additive c-club $\mathcal{C}$ has the following property:
$\langle \bigcup\{\mathcal{A}_n : 0 \leq n\} \rangle \in \mathcal{C}$ for any $\{\mathcal{A}_n : 0 \leq n\} \subset \mathcal{C}$.
To see this, first check inductively that $\langle \bigcup\{A_k : k \leq n\} \rangle \in \mathcal{C}$, and then use (i).

Recent result about skeletally Dugundji space

Theorem

For a Tychonoff space X the following conditions are equivalent:

1. *X is a skeletally Dugundji space;*
2. *X has an ω-multiplicative lattice of skeletal maps;*
3. *X has a multiplicative lattice of skeletal maps;*
4. *There exists an additive c-club $\mathcal{C}$;*
5. *There exists an additive c-club $\mathcal{C}$ such that $\langle \bigcup \mathcal{R} \rangle \subset_c coZ(X)$ for any family $\mathcal{R} \subseteq \mathcal{C}$.*

Proof.

Again, see the article by A. Kucharski, Sz. Plewik and V. Valov (2015), which is down-loadable from the arXiv:1501.06734. □

Additive d-clubs

We say that a space X is *d-openly generated* if and only if the family

$$\{\mathcal{P} \in [coZ(X)]^\omega : \mathcal{P} \subset_! coZ(X)\}$$

contains a club. Here, $\mathcal{P} \subset_! coZ(X)$ means that for any $\mathcal{S} \subset \mathcal{P}$ and $x \notin \overline{\bigcup \mathcal{S}}$, there exists $W \in \mathcal{P}$ such that $x \in W$ and $W \cap \bigcup \mathcal{S} = \varnothing$.

Since $\mathcal{P} \subset_! coZ(X)$ implies $\mathcal{P} \subset_c coZ(X)$, a c-club $\mathcal{C}$ is called a *d-club* provided for each $\mathcal{P} \in \mathcal{C}$ satisfies $\mathcal{P} \subset_! coZ(X)$. Still so each $P \in \mathcal{C}$ has the property Seq and P is closed under finite unions and finite intersections.

As in the definition of c-clubs, a d-club $\mathcal{C}$ is said to be *additive d-club*, whenever $\langle \mathcal{A}_1 \cup \mathcal{A}_2 \rangle \in \mathcal{C}$ for all $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{C}$. Also, each additive d-club $\mathcal{C}$ satisfies:

$$\{\mathcal{A}_n : 0 \leq n\} \subset \mathcal{C} \Rightarrow \left\langle \bigcup \{\mathcal{A}_n : 0 \leq n\} \right\rangle \in \mathcal{C}.$$

Results with open and d-open maps

Here is the characterization of spaces with lattices of d-open maps:

Theorem

For a Tychonoff space X the following conditions are equivalent:

(1) *X has an ω-multiplicative lattice of d-open maps;*

(2) *There exists an additive d-club for X;*

(3) *X has a multiplicative lattice of d-open maps.*

And also a the characterization of Dugundji spaces:

Theorem

A compact Hausdorff space X is a Dugundji space if and only if there exists an additive d-club for X.

Some final remarks

In fact, a proof of the above two theorem follows the same pattern as for skeletally Dugundji spaces. Only small corrections are sufficient to replace completely embeddings, i.e., change $\mathcal{P} \subset_c coZ(X)$ onto $\mathcal{P} \subset_! coZ(X)$, and the following observation:

Remark

For any map $f : X \to Y$ the following conditions are equivalent:

(1) *f is d-open;*

(2) *There is a base $\mathcal{B}$ of Y such that $\{f^{-1}(V) : V \in \mathcal{B}\} \subset_! coZ(X)$;*

(3) *$\{f^{-1}(V) : V \in \mathcal{B}\} \subset_! coZ(X)$ for every base $\mathcal{B}$ of Y.*

Some final remarks -continuation

Remark

For any map $f : X \to Y$ the following conditions are equivalent:

(1) *f is skeletal;*

(2) *There is a π-base $\mathcal{B}$ of Y such that $\{f^{-1}(V) : V \in \mathcal{B}\} \subset_c coZ(X)$;*

(3) *$\{f^{-1}(V) : V \in \mathcal{B}\} \subset_c coZ(X)$ for every π-base $\mathcal{B}$ of Y.*

Remark

For any map $f : X \to Y$ between compact Hausdorff spaces the following conditions are equivalent:

(1) *f is open;*

(2) *There is a base $\mathcal{B}$ of Y such that $\{f^{-1}(V) : V \in \mathcal{B}\} \subset_! coZ(X)$;*

(3) *$\{f^{-1}(V) : V \in \mathcal{B}\} \subset_! coZ(X)$ for every base $\mathcal{B}$ of Y.*

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